Journal of the Egyptian Mathematical Society (2015) 23, 247–250

Egyptian Mathematical Society
Journal of the Egyptian Mathematical Society
www.etms-eg.org
www.elsevier.com/locate/joems

ORIGINAL ARTICLE

A new relation including ${}_2F_2$ between Laguerre and Hermite matrix polynomials

Bayram Çekim *

Gazi University, Faculty of Science, Department of Mathematics, 06500 Teknik Okullar-Ankara, Turkey

Received 30 January 2014; revised 22 April 2014; accepted 6 May 2014
Available online 14 June 2014

KEYWORDS

Hermite matrix polynomials;
Laguerre matrix polynomials;
Hypergeometric matrix functions

Abstract In the present paper, a new relation including hypergeometric matrix function between Laguerre and Hermite matrix polynomials presented in [2,3] is derived.

2010 MATHEMATICS SUBJECT CLASSIFICATION: Primary 33C25, 33D15; Secondary 15A60

1. Introduction

Theory of orthogonal matrix polynomials is a growing field of applied mathematics which owes its development from theoretical and practical examples given below. The property of orthogonality [1–3], Rodrigues formula [1], a second-order Sturm–Liouville differential equation [1], a three-term matrix recurrence formula [4], relation between different orthogonal matrix polynomials [5] and matrix polynomials of several variables [6,18–20] are theoretical examples for orthogonal matrix polynomials. Besides, the practical examples for matrix polynomials can be seen in statistics, group representation theory [7], scattering theory [8], differential equations [3], Fourier series expansions [9], quadrature [10], splines [11] and medical imaging [12].

The aim of this paper is to derive a connection between Laguerre and Hermite matrix polynomials recently presented in [2,3].

Now let us give some known facts and definitions.

If A is a matrix in $\mathbb{C}^{r\times r}$, we denote by $\sigma(A)$ the set of all the eigenvalues of A. If $f(z), g(z)$ are holomorphic functions in an open set Ω of the complex plane, and if $\sigma(A) \subset \mathbb{C}$, we denote by $f(A), g(A)$, respectively, the image by the Riesz–Dunford functional calculus of the functions $f(z), g(z)$, respectively, acting on the matrix A, and

$$f(A)g(A) = g(A)f(A)$$

see [13]. The two-norm of A, which will be denoted by $\|A\|$, is defined by

$$\|A\| = \sup_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2},$$

where, for a vector $y \in \mathbb{C}^N$, $\|y\|_2 = (y^T y)^{1/2}$ is the Euclidean norm of y. For $A, B \in \mathbb{C}^{r\times r}$, this norm also satisfies the following properties

$$\begin{aligned} \|A+B\| &\leqslant \|A\| + \|B\| \\ \|AB\| &\leqslant \|A\|\|B\|. \end{aligned} \tag{1}$$

* Tel.: +90 3122021084.
E-mail address: bayramcekim@gazi.edu.tr.
Peer review under responsibility of Egyptian Mathematical Society.

Production and hosting by Elsevier

1110-256X © 2014 Production and hosting by Elsevier B.V. on behalf of Egyptian Mathematical Society.
http://dx.doi.org/10.1016/j.joems.2014.05.005

Throughout this paper, a matrix polynomial of degree n means an expression of the form

$$P(x) = A_n x^n + A_{n-1}x^{n-1} + \ldots + A_1 x + A_0,$$

where x is a real variable and A_j $(0 < j < n)$ are $r \times r$ complex matrices. For any matrix A in $\mathbb{C}^{r\times r}$, we denote Pochhammer symbol:

$$(A)_n = A(A+I)\ldots(A+(n-1)I), \quad n \geqslant 1, \ (A)_0 = I. \tag{2}$$

The hypergeometric matrix function $F(A,B;C;z)$ has been given in the form [14]:

$$F(A,B;C;z) = \sum_{n=0}^{\infty} \frac{(A)_n (B)_n}{n!} \left[(C)_n\right]^{-1} z^n \tag{3}$$

for matrices A, B and C in $\mathbb{C}^{r\times r}$ such that $C + nI$ is invertible for all integer $n \geqslant 0$ and for $|z| < 1$. In [14], Defez and Jódar show that for matrices $A(k,n)$ and $B(k,n)$ in $\mathbb{C}^{r\times r}$ where $n \geqslant 0, k \geqslant 0$, the following relations are satisfied

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} A(k,n-k) \tag{4}$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n+k) \tag{5}$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\left[\frac{n}{2}\right]} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n+2k). \tag{6}$$

2. Known properties of Laguerre and Hermite matrix polynomials

For the sake of clarity in the presentation, we recall that if A is a matrix in $\mathbb{C}^{r\times r}$ such that

$$-k \text{ is not an eigenvalue of } A \text{ for every integer } k > 0 \tag{7}$$

and λ is a complex number with Re $\lambda > 0$, then the n-th Laguerre matrix polynomial is defined by [3]

$$L_n^{(A,\lambda)}(x) = \sum_{k=0}^{n} \frac{(-1)^k \lambda^k}{k!(n-k)!}(A+I)_n (A+I)_k^{-1} x^k; \ n \geqslant 0.$$

Furthermore, the following explicit formula holds:

$$x^n I = \sum_{k=0}^{n} \frac{(-1)^k \lambda^{-n} n! (A+I)_n [(A+I)_k]^{-1}}{(n-k)!} L_k^{(A,\lambda)}(x) \tag{8}$$

see [15]. For the definition of Hermite matrix polynomials, let us suppose that A is a matrix such that

$$\text{Re } z > 0 \text{ for every eigenvalue } z \in \sigma(A) \tag{9}$$

and let us denote $\sqrt{A} = exp((1/2)\ \log A)$ by the image of the function $z^{\frac{1}{2}} = \exp((1/2)\log z)$ by the Riesz–Dunford functional calculus, acting on the matrix A, where $\log z$ denotes the principal branch of the complex logarithm. Then by [2] the n – Hermite matrix polynomial $H_n(x,A)$ is defined by

$$H_n(x,A) = n! \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k}{k!(n-2k)!}\left(x\sqrt{2A}\right)^{n-2k} \ ; \ n \geqslant 0.$$

Furthermore, the following generating matrix function formula for these matrix polynomials holds:

$$\sum_{n=0}^{\infty} \frac{H_n(x,A)}{n!} t^n = \exp\left(xt\sqrt{2A} - It^2\right). \tag{10}$$

In addition to these facts, the interesting connection between Laguerre and Hermite matrix polynomials is given by [5]

$$\begin{aligned} &\frac{(-1)^n}{\sqrt{\pi}(2n)!}\Gamma(A+(n+1)I)\Gamma^{-1}\left(A+\frac{1}{2}I\right) \\ &\quad\times \int_{-1}^{1}(1-t^2)^{A-\frac{1}{2}I} H_{2n}(t\sqrt{x},A)dt \\ &\quad= \sum_{k=0}^{n}\frac{x^k}{k!}\left(\lambda I - \frac{1}{2}A\right)^k L_{n-k}^{(A+kI,\lambda)}(x) \ ; \ n \geqslant 0, x > 0. \end{aligned}$$

3. A new relation between these matrix polynomials

Let's give a definition of ${}_2F_2$ as generalization of the hypergeometric matrix function.

Definition 1. According to presentation of the hypergeometric function, ${}_2F_2$ hypergeometric matrix function is defined as

$${}_2F_2\left(\begin{matrix} A, & B \\ C, & D \end{matrix} ; x\right) = \sum_{n=0}^{\infty} \frac{(A)_n (B)_n}{n!} (C)_n^{-1} (D)_n^{-1} x^n \tag{11}$$

for matrices A, B, C, D in $\mathbb{C}^{r\times r}$ such that $C + nI$ and $D + nI$ are invertible for all integer $n \geqslant 0$.

Now that find values x for which this series (11) is convergent.

Let's write

$$\left.\begin{aligned} (C+nI)^{-1} &= \tfrac{1}{n}\left(\tfrac{C}{n}+I\right)^{-1} \\ (D+nI)^{-1} &= \tfrac{1}{n}\left(\tfrac{D}{n}+I\right)^{-1} \end{aligned}\right\}. \tag{12}$$

If $n > \|C\|$ and $n > \|D\|$, due to perturbation Lemma [13], it follows

$$\left.\begin{aligned} \left\|\left(\tfrac{C}{n}+I\right)^{-1}\right\| &\leqslant \tfrac{1}{1-\frac{\|C\|}{n}} = \tfrac{n}{n-\|C\|} \\ \left\|\left(\tfrac{D}{n}+I\right)^{-1}\right\| &\leqslant \tfrac{1}{1-\frac{\|D\|}{n}} = \tfrac{n}{n-\|D\|} \end{aligned}\right\}. \tag{13}$$

Let $x \neq 0$ be a complex number and let's consider the expression

$$\frac{\|(A)_{n+1}\|\|(B)_{n+1}\|\|(C)_{n+1}^{-1}\|\|(D)_{n+1}^{-1}\|n!}{\|(A)_n\|\|(B)_n\|\|(C)_n^{-1}\|\|(D)_n^{-1}\|(n+1)!}\frac{|x|^{n+1}}{|x|^n}. \tag{14}$$

Using Pochhammer symbol (2) and (1) in (14), one gets

$$\begin{aligned} &\frac{\|(A)_{n+1}\|\|(B)_{n+1}\|\|(C)_{n+1}^{-1}\|\|(D)_{n+1}^{-1}\|n!}{\|(A)_n\|\|(B)_n\|\|(C)_n^{-1}\|\|(D)_n^{-1}\|(n+1)!}\frac{|x|^{n+1}}{|x|^n} \\ &\quad\leqslant \frac{\|A\|_{n+1}\|B\|_{n+1}\|(C)_{n+1}^{-1}\|\|(D)_{n+1}^{-1}\|}{\|A\|_n\|B\|_n\|(C)_n^{-1}\|\|(D)_n^{-1}\|}\frac{|x|}{n+1}. \end{aligned} \tag{15}$$

For $n > \|C\|$ and $n > \|D\|$, by (1), (13) and (15) can be written

$$\begin{aligned} &\frac{\|(A)_{n+1}\|\|(B)_{n+1}\|\|(C)_{n+1}^{-1}\|\|(D)_{n+1}^{-1}\|n!}{\|(A)_n\|\|(B)_n\|\|(C)_n^{-1}\|\|(D)_n^{-1}\|(n+1)!}\frac{|x|^{n+1}}{|x|^n} \\ &\quad\leqslant (\|A\|+n)(\|B\|+n)\frac{1}{n-\|C\|}\frac{1}{n-\|D\|}\frac{|x|}{n+1} \to 0 \ \text{ as } n\to\infty \end{aligned}$$

By ratio test, ${}_2F_2$ hypergeometric matrix series is convergent for any complex number x. Let's start now to show the connection satisfied by Laguerre and Hermite matrix polynomials.

For the principal square root of I(see [17]), by the generating matrix function (10), one can write

$$\sum_{n=0}^{\infty} \frac{H_n\left(x, \frac{I}{2}\right)}{n!} t^n = \exp\left[xtI - t^2 I\right]. \tag{16}$$

Substituting following expression

$$\exp\left[xtI - t^2 I\right] = \exp\left[xtI\right] \exp(-t^2 I)$$

and using Taylor series of functions in the right-hand side of above equation, one gets

$$\exp[xtI - t^2 I] = \sum_{n,s=0}^{\infty} \frac{(-1)^s x^n}{n!\ s!} t^{n+2s} I.$$

From (8), it follows that

$$\exp[xtI - t^2 I] = \sum_{n,s=0}^{\infty} \frac{(-1)^s}{n!\ s!} \times \left\{ \sum_{k=0}^{n} \frac{(-1)^k n! \lambda^{-n} (A+I)_n [(A+I)_k]^{-1}}{(n-k)!} L_k^{(A,\lambda)}(x) \right\} t^{n+2s}. \tag{17}$$

Using (5) and (17) can be written in the form

$$\exp[xtI - t^2 I] = \sum_{n,s,k=0}^{\infty} \frac{(-1)^{s+k}}{n!\ s!} \lambda^{-n-k} (A+I)_{n+k} [(A+I)_k]^{-1} \times L_k^{(A,\lambda)}(x) t^{n+k+2s}.$$

By (6), one gets

$$\exp[xtI - t^2 I] = \sum_{n,k=0}^{\infty} \sum_{s=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^{s+k}}{(n-2s)!\ s!} \lambda^{-n+2s-k} (A+I)_{n-2s+k} \times [(A+I)_k]^{-1} L_k^{(A,\lambda)}(x) t^{n+k}. \tag{18}$$

From Pochhammer symbol in (2), one can write following equations

$$\frac{1}{(n-2s)!} I = \frac{1}{n!} 2^{2s} \left(\frac{-n}{2} I\right)_s \left(\frac{-(n-1)}{2} I\right)_s$$

$$(A+I)_{n-2s+k} = (A+I)_{n+k} \left[\left(\frac{-1}{2}(A+(n+k)I)\right)_s \times \left(\frac{-1}{2}(A+(n+k-1)I)\right)_s 2^{2s} \right]^{-1}.$$

Writing these equations in (18), one have

$$\exp[xtI - t^2 I] = \sum_{n,k=0}^{\infty} \frac{(-1)^k}{n!} \lambda^{-n-k} (A+I)_{n+k} (A+I)_k^{-1} L_k^{(A,\lambda)}(x) \times \sum_{s=0}^{\left[\frac{n}{2}\right]} \left\{ \left(\frac{-n}{2}\right)_s \left(\frac{-(n-1)}{2}\right)_s \times \left[\left(\frac{-1}{2}(A+(n+k)I)\right)_s \times \left(\frac{-1}{2}(A+(n+k-1)I)\right)_s \right]^{-1} \frac{(-1)^s}{s!} \lambda^{2s} \right\} t^{n+k}.$$

Thus, one can write

$$\exp[xtI - t^2 I] = \sum_{n,k=0}^{\infty} \left\{ \frac{(-1)^k}{n!} \lambda^{-n-k} (A+I)_{n+k} (A+I)_k^{-1} L_k^{(A,\lambda)}(x) \times {}_2F_2\left(\begin{matrix} \frac{-n}{2} I, & \frac{-(n-1)}{2} I \\ \frac{-1}{2}(A+(n+k)I), & \frac{-1}{2}(A+(n+k-1)I) \end{matrix} ; -\lambda^2 \right) \right\} t^{n+k}. \tag{19}$$

In (19), by (4), it follows that

$$\exp\left[xtI - t^2 I\right] = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \left\{ \frac{(-1)^k}{(n-k)!} \lambda^{-n} (A+I)_n (A+I)_k^{-1} L_k^{(A,\lambda)}(x) \times {}_2F_2\left(\begin{matrix} \frac{-(n-k)}{2} I, & \frac{-(n-k-1)}{2} I \\ \frac{-1}{2}(A+nI), & \frac{-1}{2}(A+(n-1)I) \end{matrix} ; -\lambda^2 \right) \right\} t^n. \tag{20}$$

Combining (16) and (20) and comparing coefficients of t^n, we have the following desired relation

$$H_n\left(x, \frac{I}{2}\right) = \sum_{k=0}^{n} \lambda^{-n} (-n)_k (A+I)_n\ (A+I)_k^{-1} \times {}_2F_2\left(\begin{matrix} \frac{-(n-k)}{2} I, & \frac{-(n-k-1)}{2} I \\ \frac{-1}{2}(A+nI), & \frac{-1}{2}(A+(n-1)I) \end{matrix} ; -\lambda^2 \right) L_k^{(A,\lambda)}(x)$$

Thus the result has been established:

Theorem 1. *For the principal square root of I, if A is a matrix in $\mathbb{C}^{r\times r}$ satisfying (7) and λ is a complex number $Re\ (\lambda) > 0$, Laguerre and Hermite matrix polynomials satisfy*

$$H_n\left(x, \frac{I}{2}\right) = \sum_{k=0}^{n} \lambda^{-n} (-n)_k (A+I)_n\ (A+I)_k^{-1} \times {}_2F_2\left(\begin{matrix} \frac{-(n-k)}{2} I, & \frac{-(n-k-1)}{2} I \\ \frac{-1}{2}(A+nI), & \frac{-1}{2}(A+(n-1)I) \end{matrix} ; -\lambda^2 \right) \times L_k^{(A,\lambda)}(x).$$

For the special case of $r = 1$, taking $A = \alpha,\ \lambda = \frac{1}{2},\ x \to 2x$, the above equation reduces to the known relation between Hermite and Laguerre polynomials (see [16]).

Acknowledgment

The author would like to thank referees for their valuable comments and suggestions, which have improved the quality of the paper.

References

[1] E. Defez, L. Jódar, Chebyshev matrix polynomials and second order matrix differential equations, Util. Math. 61 (2002) 107–123.

[2] L. Jódar, R. Company, Hermite matrix polynomials and second order matrix differential equations, J. Approx. Theory Appl. 12 (2) (1996) 20–30.

[3] L. Jódar, R. Company, E. Navarro, Laguerre matrix polynomials and system of second-order differential equations, Appl. Numer. Math. 15 (1994) 53–63.

[4] E. Defez, L. Jódar, A. Law, E. Ponsoda, Three-term recurrences and matrix orthogonal polynomials, Util. Math. 57 (2000) 129–146.

[5] L. Jódar, E. Defez, A connection between Laguerre's and Hermite's matrix polynomials, Appl. Math. Lett. 11 (1) (1998) 13–17.

[6] E. Erkuş-Duman, Matrix extensions of polynomials in several variables, Util. Math. 85 (2011) 161–180.

[7] A.T. James, Special functions of matrix and single argument in statistics, in: R.A. Askey (Ed.), Theory and Applications of Special Functions, Academic Press, 1975, pp. 497–520.

[8] J.S. Geronimo, Scattering theory and matrix orthogonal polynomials on the real line, Circ. Syst. Signal Process 1 (3–4) (1982) 471–494.

[9] E. Defez, L. Jódar, Some applications of the Hermite matrix polynomials series expansions, J. Comput. Appl. Math. 99 (1998) 105–117.

[10] L. Jódar, E. Defez, E. Ponsoda, Matrix quadrature and orthogonal matrix polynomials, Congr. Numer. 106 (1995) 141–153.

[11] E. Defez, A. Law, J. Villanueva-Oller, R.J. Villanueva, Matrix cubic splines for progressive 3D imaging, J. Math. Imag. Vision 17 (2002) 41–53.

[12] E. Defez, A. Hervás, A. Law, J. Villanueva-Oller, R.J. Villanueva, Progressive transmission of images: PC-based computations, using orthogonal matrix polynomials, Math. Comput. Modelling 32 (2000) 1125–1140.

[13] N. Dunford, J. Schwartz, Linear Operators, Part I, Wiley Interscience, New York, 1957.

[14] L. Jódar, J.C. Cortés, On the hypergeometric matrix function, J. Comput. Appl. Math. 99 (1998) 205–217.

[15] L. Jódar, J. Sastre, On Laguerre matrix polynomials, Util. Math. 53 (1998) 37–84.

[16] E.D. Rainville, Special Functions, Chelsea Pub., 1971.

[17] N.J. Higham, Functions of Matrices: Theory and Computation, Siam, 2008.

[18] R. Aktaş, A new multivariable extension of Humbert matrix polynomials, in: AIP Conference Proceedings, vol. 1558, 2013, pp. 1128–1131.

[19] R. Aktaş, B. Çekim, R. Şahin, The matrix version for the multivariable Humbert polynomials, Miskolc Math. Notes 13 (2) (2012) 197–208.

[20] F. Taşdelen, B. Çekim, R. Aktaş, On a multivariable extension of Jacobi matrix polynomials, Comput. Math. Appl. 61 (2011) 2412–2423.